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The Integral Structure of Some Bounded Operators on $\mathbf{L}_1(\mu, X)$

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Abstract. Let $(S, \mathcal{F}, \mu)$ be a finite measure space and let X be a Banach space. As usual $L_1(\mu, X)$ is the Banach space of all Bochner μ-integrable functions $f : S \to X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. This work is intended for the study of a class of linear bounded operators $T : L_1(\mu, X) \to X$, whose integral structure is much similar to that of bounded functionals on $L_1(\mu)$. We give two complete characterizations of this class. The first one, which may be considered as a Riesz type theorem, is obtained via integrals by functions in $L_\infty(\mu)$. Actually the identified class is isometrically isomorphic to $L_\infty(\mu)$. The second characterization is more specific. It pertains to an operator valued measure, that will be attached to each operator of the class. This operator valued measure will be absolutely continuous with respect to μ and this property will be used to get another interesting characterization of the class under consideration.

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1 Introduction

1.1. Let $(S, \mathcal{F}, \mu)$ be a finite measure space and let X be a Banach space. We denote by $L_1(\mu, X)$ the Banach space of all Bochner μ-integrable functions $f : S \to X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. For all properties of the Bochner integral, we refer the reader to [2].
For $f \in L_1(\mu, X)$, we put:

$$(1.2) \qquad \|f\|_1 = \int_S \|f(s)\| \, d\mu(s)$$

Then it is well known that:

1.3. Proposition: *Formula (1.2) defines a norm on $L_1(\mu, X)$, for which $L_1(\mu, X)$ is a Banach space. Moreover the measurable simple functions $s : S \to X$ form a dense subspace of $L_1(\mu, X)$. This means that for each $f \in L_1(\mu, X)$ there is a sequence s_n of simple functions such that $\|f - s_n\|_1 \to 0$.*

The starting point that has motivated the present work is contained in the following simple observation:

1.4. Theorem: *Fix a function g in $L_\infty(\mu)$ (the space of all $\mu-$essentially bounded real functions on S) and consider the operator $T_g : L_1(\mu, X) \to X$ defined by:*

$$(1.5) \qquad f \in L_1(\mu, X), \qquad T_g(f) = \int_S fg \, d\mu:$$

Then T_g is linear bounded and satisfies $\|T_g\| = \|g\|_\infty$.

Proof: Since $\|f(s)g(s)\| \leq \|f(s)\| \, \|g\|_\infty$ $\mu-$ a.e. we deduce from (1.5), $\|T_g(f)\| \leq \|g\|_\infty \cdot \int_S \|f(s)\| \, d\mu(s) = \|g\|_\infty \cdot \|f\|_1$. So the operator T_g is bounded and $\|T_g\| \leq \|g\|_\infty$. To prove the reverse inequality, apply T_g to a function $f \in L_1(\mu, X)$ of the form $f = \varphi.x$, where $\varphi \in L_1(\mu)$, such that $\|\varphi\|_1 = 1$ and x fixed in X with $\|x\| = 1$. We get $\|f\|_1 = \|\varphi\|_1 = 1$ and $T_g(f) = \int_S \varphi g \, x d\mu = \left(\int_S \varphi g.d\mu\right).x$, by standard integration tools. So we deduce $\|T_g(f)\| = \left|\int_S \varphi g.d\mu\right| \leq \|T_g\|$ and then $Sup\left\{\left|\int_S \varphi g.d\mu\right|, \, \varphi \in L_1(\mu), \, \|\varphi\|_1 = 1\right\} \leq \|T_g\|$. But the LHS of the preceding inequality is equal to $\|g\|_\infty$ by the Riesz duality theorem for $L_1(\mu)$. So we get $\|g\|_\infty \leq \|T_g\|$ and then $\|T_g\| = \|g\|_\infty$.■

1.6. Remark: Another way to put the conclusion of theorem 1.4 is the following:

The map $\Phi : g \to T_g$ from $L_\infty(\mu)$ into $\mathcal{L}(L_1(\mu, X), X)$, the space of bounded operators $T : L_1(\mu, X) \to X$, is a linear isometry.

We can wonder whether Φ is onto. This is certainly true if $X = \mathbb{R}$ by the Riesz duality theorem for $L_1(\mu)$. But if dimension of X is greater than one, the following example shows that not all operators in $\mathcal{L}(L_1(\mu, X), X)$ can be written in the form (1.5) for some g in $L_\infty(\mu)$.

1.7. Example: Let $X = \mathbb{R}^2$, equipped with the norm: $z = (z_1, z_2)$, $\|z\| = |z_1| + |z_2|$. If $f = (f_1, f_2) : S \to \mathbb{R}^2$ is Bochner $\mu-$integrable with the Borel $\sigma-$field on $\mathbb{R}^2$, then $f_1, f_2 : S \to \mathbb{R}$ are $\mu-$integrable and $\int_S f \, d\mu = \left(\int_S f_1 \, d\mu, \int_S f_2 \, d\mu\right)$. Note also that $\|f(s)\| = |f_1(s)| + |f_2(s)|$, so that $\|f\|_1 = \int_S |f_1| \, d\mu + \int_S |f_2| \, d\mu$. Now define the operator $T : L_1(\mu, \mathbb{R}^2) \to \mathbb{R}^2$, by $Tf = T(f_1, f_2) = \left(\int_S f_1 \, d\mu, \alpha \int_S f_2 \, d\mu\right)$, where $0 < \alpha < 1$ is a fixed constant. It is clear that T is linear and we have $\|Tf\| = \left|\int_S f_1 \, d\mu\right| + \alpha \left|\int_S f_2 \, d\mu\right| \leq \|f\|_1$, so that T is bounded. If there were a $g \in L_\infty(\mu)$ such that $T(f) = \int_S fg \, d\mu$, we would have $\int_S f_1 \, d\mu = \int_S f_1 \, g.d\mu$ and $\alpha \int_S f_2 \, d\mu = \int_S f_2.g.d\mu$, for all

$\mu-$integrable functions f_1, f_2. Taking f_1, f_2 both characteristic functions of sets in $\mathcal{F}$, this would imply $g = 1$, $\mu-$a.e and $g = \alpha$, $\mu-$a.e. This is impossible by the choice of α. Consequently the operator T cannot be written in the form (1.5).

The aim of this work is to characterize bounded operators $T : L_1(\mu, X) \to X$ that have integral form (1.5) with a function $g \in L_\infty(\mu)$. This amounts to describe the range of the operator Φ. In section 2 we give the ingredients of this characterization. These ingredients are similar to those given in [3] for operators on the space of continuous functions. This allows a representation of operators on the space $L_1(\mu, X)$, much simpler than those given for the space $L_1(\mu)$ [1], or more recently [4], [5], for the space $L_1(\mu, X)$ itself. In section 3 we prove integral representation by operator valued measures, for operators introduced in section 2. This leads to a rather precise description of such operators.

2 The Characterization

In this section we want to identify those operators $T \in \mathcal{L}(L_1(\mu, X), X)$, for which there is $g \in L_\infty(\mu)$ such that $T = T_g$. To this end we shall extend the strategy used in [] to the present setting. Let X^* be the topological dual of X. For each $x^* \in X^*$ consider the operator $\varphi_{x^*} : L_1(\mu, X) \to L_1(\mu)$, given by:

$$(2.1) \qquad f \in L_1(\mu, X), \quad \varphi_{x^*} f = x^* \circ f$$

where $(x^* \circ f)(t) = x^*(f(t))$, $t \in S$.
We collect some facts about φ_{x^*} for later use:

2.2. Proposition: *(a)* φ_{x^*} *is linear bounded and* $\|\varphi_{x^*}\| = \|x^*\|$.
(b) φ_{x^*} *is onto for each* $x^* \neq 0$.
(c) There exist $y^* \in X^*$ *such that for each* $h \in L_1(\mu)$ *there is* $f \in L_1(\mu, X)$ *with*
$\|f\|_1 = \|h\|_1$ *and* $\varphi_{y^*} f = h$.

Proof: *(a)* $\|\varphi_{x^*} f\| = \int_S |x^* \circ f|\, d\mu \leq \|x^*\| \int_S \|f(s)\|\, d\mu(s) = \|x^*\| \|f\|_1$. So φ_{x^*} is bounded and $\|\varphi_{x^*}\| \leq \|x^*\|$. To see the reverse inequality apply φ_{x^*} to a function $f \in L_1(\mu, X)$ of the form $f(\bullet) = g(\bullet).x$, with $g \in L_1(\mu)$ such that $\|g\|_1 = 1$ and x fixed in X with $\|x\| = 1$. We get $\|f\|_1 = 1$ and $\|\varphi_{x^*} f\| = \int_S |x^* \circ f|\, d\mu = |x^*(x)|$. Thus $|x^*(x)| \leq \|\varphi_{x^*}\|$ for every $x \in X$ with $\|x\| = 1$. Consequently $\|x^*\| = Sup\{|x^*(x)|, x \in X, \ \|x\|_1 = 1\} \leq \|\varphi_{x^*}\|$.
(b) Let $x^* \neq 0$ and choose $x \in X$ such that $x^*(x) = 1$. Now if $h \in L_1(\mu)$ put $f = h.x$, then clearly we have $\varphi_{x^*} f = h$..
(c) Choose $x \in X$ with $\|x\| = 1$, then (Hahn-Banach theorem) choose $y^* \in X^*$ such that $y^*(x) = \|x\| = 1$, $\|y^*\| = 1$. If $h \in L_1(\mu)$, the function $f = h.x$ is in $L_1(\mu, X)$ and fits the conclusion.■

The following class of operators will play an essential role for the characterization we need:

2.3. Definition: Let $\mathfrak{D}$ be the class of linear bounded operators $T \in \mathcal{L}(L_1(\mu, X), X)$ satisfying the following condition:

$$x^*, y^* \in X^*,\ f, g \in L_1(\mu, X) : \varphi_{x^*} f = \varphi_{y^*} g \Longrightarrow x^* T f = y^* T g \qquad (2.4)$$

It is easy to check that $\mathfrak{D}$ is a closed subspace of $\mathcal{L}(L_1(\mu, X), X)$. Note also that every T_g as defined by (1.5) is in $\mathfrak{D}$.
The important fact about $\mathfrak{D}$ is:

2.5. Theorem: *Let T be an operator in $\mathfrak{D}$, then there exists a unique bounded linear functional $V : L_1(\mu) \to \mathbb{R}$ such that:*

$$V \circ \varphi_{x^*} = x^* \circ T \qquad (2.6)$$

for every $x^* \in X^*$.

Proof: Let $h \in L_1(\mu)$ and $x^* \in X^*$, $x^* \neq 0$; by 2.2(b) there is an $f \in L_1(\mu, X)$ such that $\varphi_{x^*} f = h$. then we put:

$$V(h) = x^* T f \qquad (2.7)$$

If $\varphi_{x^*} f = \varphi_{y^*} g = h$, then $x^* T f = y^* T g$, by condition (2.4); so V is well defined, and it is easy to see that it is linear. We must show that V is bounded. We may argue as follows: since φ_{x^*} is bounded and onto, by the open mapping principle there exists a constant $K = K_{x^*} > 0$ such that for every $h \in L_1(\mu)$, there is a solution $f \in L_1(\mu, X)$ of $\varphi_{x^*} f = h$, with $\|f\| \leq K. \|h\|$. From (2.7) we deduce that $\|V(h)\| \leq \|x^*\| \|T\| \|f\| \leq \|x^*\| \|T\| K \|h\|$, which proves that V is bounded.
It is noteworthy that the functional V does not depend on the choice of x^* but depends only on T. For if V_{x^*} and V_{y^*} are defined as in (2.7), with $x^*, y^* \neq 0$, then $V_{x^*}(h) = x^* T f$ if $h = \varphi_{x^*} f$ and $V_{y^*}(h) = y^* T g$ if $h = \varphi_{y^*} g$; but condition (2.4) on T implies that $V_{x^*}(h) = V_{y^*}(h)$. It remains to prove (2.6). For $f \in L_1(\mu, X)$ and $x^* \in X^*$, we have $h = \varphi_{x^*} f \in L_1(\mu)$, and (2.7) gives $V(h) = V(\varphi_{x^*} f) = x^* T f$. Since f and x^* are arbitrary, (2.6) follows. Uniqueness is clear from (2.6) since φ_{x^*} is onto.■

As a consequence of the preceding theorem let us note:

2.8. Theorem: *There is an isometric isomorphism between the Banach space $\mathfrak{D}$ and the topological dual $L_1^*(\mu)$ of $L_1(\mu)$, for each non trivial Banach space X.*

Proof: Define the operator $\Psi : \mathfrak{D} \to L_1^*(\mu)$ by: $T \in \mathfrak{D}$, $\Psi(T) = V$, where V is the unique bounded functional on $L_1(\mu)$ attached to T by theorem 2.5. It is not difficult to see that Ψ is linear. We have to show that Ψ is an isometry, that is, $\|V\| = \|T\|$ if $\Psi(T) = V$. First we prove the estimation

$$\|V\| = Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \tag{2.9}$$

We have $\|V \circ \varphi_{x^*}\| \leq \|V\| \|\varphi_{x^*}\| = \|V\| \|x^*\|$, since $\|\varphi_{x^*}\| = \|x^*\|$ by 2.2(a). So we deduce $\|V \circ \varphi_{x^*}\| \leq \|V\|$, for all $x^* \in X^*$, with $\|x^*\| \leq 1$. Hence $Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \leq \|V\|$. But $V \in L_1^*(\mu)$, consequently for each $\varepsilon > 0$ there is $h \in L_1(\mu)$ such that $\|h\|_1 \leq 1$ and $\|V\| - \varepsilon < |V(h)| \leq \|V\|$. Now let $y^* \in X^*$ as in 2.2 (c) and choose $f \in L_1(\mu, X)$, such that $\|f\|_1 = \|h\|_1$ and $\varphi_{y^*} f = h$. Then $\|f\|_1 \leq 1$ and $|V \circ \varphi_{y^*}(f)| = |V(h)| \leq \|V \circ \varphi_{y^*}\| \|f\|_1$. Thus $|V(h)| \leq \|V \circ \varphi_{y^*}\| \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. From the choice of h we get $\|V\| - \varepsilon \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. Letting $\varepsilon \downarrow 0$, we obtain $\|V\| \leq Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. So (2.9) is proved. To finish the norm equality $\|V\| = \|T\|$, we appeal to formula (2.6) and conclude:

$\|V\| = Sup\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$
$= Sup\{\|x^* \circ T\| : x^* \in X^*, \|x^*\| \leq 1\} = \|T\|$.

To achieve the proof it remains to prove that Ψ is onto. If $V \in L_1^*(\mu)$, then by the Riesz duality theorem, there is a unique $g \in L_\infty(\mu)$ such that $V(h) = \int_S hg\, d\mu$, for all $h \in L_1(\mu)$. Consider the operator T_g on $L_1(\mu, X)$ given by formula (1.5). We have $T_g \in \mathfrak{D}$ and it is straightforward that V and T_g are linked by equation (2.6). So from the definition of the operator Ψ we deduce that $\Psi(T_g) = V$.■

Since $L_1^*(\mu)$ is isometrically isomorphic to $L_\infty(\mu)$, we deduce the following corollary:

corollary: *The class $\mathfrak{D}$ is isometrically isomorphic to $L_\infty(\mu)$. In other words, a bounded operator $T : L_1(\mu, X) \to X$ is in $\mathfrak{D}$ iff there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$ and in this case $\|T\| = \|g\|_\infty$.*

Now we turn to another description of the class $\mathfrak{D}$, namely by a space of measures. This will be achieved via integrals with respect to operator valued measures.

3 Operator valued measures representing the class $\mathfrak{D}$

3.1. The integration process we shall deal with in this section is performed by an operator valued additive set function $G : \mathcal{F} \to \mathcal{L}(X, E)$, where $\mathcal{L}(X, E)$ is the space of linear bounded operators of the Banach space X into the Banach space E. The integral will be defined for measurable functions $f : S \to X$, under the assumption that G is additive and with finite semivariarion. Let us recall that semivariation means the set function $\widetilde{G}$ on $\mathcal{F}$ given by $\widetilde{G}(B) = Sup\left\|\sum_i G(A_i).x_i\right\|$, where $B \in \mathcal{F}$, and the supremum taken over all finite partitions $\{A_i\}$ of B in $\mathcal{F}$ and all finite systems of vectors $\{x_i\}$ in X, with $\|x_i\| \leq 1\ \forall i$. The function G is said to be of finite semivariation if $\widetilde{G}(B)$ is finite for all $B \in \mathcal{F}$. A simple measurable function s on S with values in the

Banach space X is a function of the form $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$, where $\{A_i\}$ is a finite partition of S in $\mathcal{F}$, and $\{x_i\}$ is a finite system of vectors in X. The symbol 1_{A_i} means the characteristic function of the set A_i. A function $f : S \to X$ is said to be measurable if there is a sequence s_n of measurable simple functions converging uniformly to f on S. If we denote by $\mathcal{I}$ and $\mathcal{M}$ the sets of simple functions and measurable functions, respectively then $\mathcal{I}$ and $\mathcal{M}$ are subspaces of the Banach space of all bounded functions $f : S \to X$, with supremum norm. Moreover $\mathcal{I}$ is dense in $\mathcal{M}$.

We define the integral of the simple function $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$ over the set $B \in \mathcal{F}$, with respect to G by:

$$(3.2) \qquad \int_B s\, dG = \sum_i G(A_i \cap B).x_i$$

It is easy to check that the integral is well defined and satisfies:

$$(3.3) \qquad \left\| \int_B s dG \right\| \leq \|s\| . \widetilde{G}(B)$$

($\|s\|$ = supremum norm)

Let us observe that estimation (3.3) implies that the linear operator $U_B : \mathcal{I} \to E$, with $U_B(s) = \int_B s\, dG$ is bounded. So we can extend it in a unique manner to a bounded operator on the closure $\mathcal{M}$ of $\mathcal{I}$. This extension will be our integration process on the space $\mathcal{M}$ of measurable functions.We shall denote it also by U_B with $U_B = U$ if $B = S$. Note that if $f \in \mathcal{M}$ and if s_n is a sequence in $\mathcal{I}$ such that $\|f - s_n\| \to 0$ then the integral of f is given by:

$$(3.4) \qquad U_B(f) = \int_B f\, dG = \lim_n \int_B s_n\, dG$$

By (3.3) the integral (3.4) does not depend on the sequence s_n chosen converging to the function f. This simple integration process will be sufficient for our purpose. The outstanding facts are summarized in the following:

3.5 Theorem: *Let G be an additive $\mathcal{L}(X, E)$-valued set function with finite semivariation on $\mathcal{F}$.Then:*
(a) The integral $\int_B f\, dG$ is linear in $f \in \mathcal{M}$ and satisfies:

$$(3.6) \qquad \widetilde{G}(B) = Sup \left\{ \left\| \int_B f\, dG \right\|, \|f\| \leq 1, \quad f \in \mathcal{M} \right\}$$

in other words the operator $U_B : \mathcal{M} \to E$ given by $U_B(f) = \int_B f\, dG$ is bounded with norm $\|U_B\| = \widetilde{G}(B)$, for each $B \in \mathcal{F}$. Conversely:
(b) Let $U : \mathcal{M} \to E$ be a bounded operator. Then there is a unique additive set function $G : \mathcal{F} \to \mathcal{L}(X, E)$, with finite semivariation such that:

$$(3.7) \qquad \forall\ f \in \mathcal{M},\ \forall\ B \in \mathcal{F}, \quad U(f.1_B) = \int_B f\, dG$$

(c) *Let $\Lambda : E \to Y$ be a bounded operator from E into the Banach space Y. Let us define $\Lambda G : \mathcal{F} \to \mathcal{L}(X,Y)$ by $(\Lambda G)(B)x = \Lambda(G(B)x)$, $B \in \mathcal{F}$, $x \in X$. Then ΛG is an additive $\mathcal{L}(X,Y)$-valued set function with finite semivariation and we have:*

$$(3.8) \qquad \forall f \in \mathcal{M}, \quad \int_S f\, d\Lambda G = \Lambda\left(\int_S f dG\right)$$

Proof: (a) To prove (3.6) start with f simple and use (3.2) and the definition of $\widetilde{G}(B)$. For general f use (3.4).
(b) Define $G : \mathcal{F} \to \mathcal{L}(X,E)$ by $G(B).x = U(1_B.x)$, for $B \in \mathcal{F}$, and $x \in X$. Then G is additive since U is linear and G is $\mathcal{L}(X,E)$-valued because U is bounded. Now (3.7) is easily checked by (3.2) and (3.4).
(c) To prove (3.8) start with f simple and use the definition of ΛG, then apply (3.4), (recall that the operator Λ is bounded).■

Actually, part (b) of this theorem is an integral representation of a bounded operator U on the space $\mathcal{M}$ by means of an $\mathcal{L}(X,E)$-valued set function G on $\mathcal{F}$.

The next step is to extend the preceding integration process from $\mathcal{M}$ to the space $L_1(\mu, X)$. The reader should observe that the space $\mathcal{M}$ is contained in $L_1(\mu, X)$, because functions in $\mathcal{M}$ are bounded and μ is a finite measure. The extension of the integral (3.4) from $\mathcal{M}$ to $L_1(\mu, X)$ will be achieved under the additional assumption that $\|G(A)\| \leq k.\mu(A)$ for some constant $k > 0$ and all $A \in \mathcal{F}$.

3.9 Theorem: *Let G be an additive $\mathcal{L}(X,E)$-valued set function with finite semivariation on $\mathcal{F}$. Assume that:*

$$(3.10) \qquad \|G(A)\| \leq k.\mu(A)$$

for some constant $k > 0$ and all $A \in \mathcal{F}$. Then we have:
(a) *The integral (3.4) is a linear operator from $\mathcal{M}$ to E which is continuous with the $L_1(\mu, X)-$topology on $\mathcal{M}$ and satisfies:*

$$(3.11) \qquad \forall f \in \mathcal{M}, \quad \left\|\int_S f dG\right\| \leq k \int_S \|f\|\, d\mu$$

(b) *The integral $\int_S f\, dG$, $f \in \mathcal{M}$, admits a unique extension to $L_1(\mu, X)$, still denoted by $\int_S f dG$, such that:*

$$(3.12) \qquad \forall f \in L_1(\mu, X), \quad \left\|\int_S f dG\right\| \leq k \int_S \|f\|\, d\mu$$

(c) *The operator $f \to \int_S f dG$ is linear and bounded from $L_1(\mu, X)$ to E.*

Proof: (a) Let $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$ be a simple measurable function with values in X. From (3.10) we deduce $\left\|\int_S s\ dG\right\| = \left\|\sum_i G(A_i).x_i\right\| \leq \sum_i \|G(A_i)\| . \|x_i\|$

$\leq \sum_i k\mu(A_i) . \|x_i\| = k \sum_i \mu(A_i) . \|x_i\| = k \int_S \|s\| \, d\mu$. So (3.11) is true for every $s \in \mathcal{I}$. Now if $f \in \mathcal{M}$, let $s_n \in \mathcal{I}$ be such that $s_n \to f$ uniformly on S. As μ is finite we deduce that $\int_S \|f - s_n\| \, d\mu \to 0$ and so $\int_S \|s_n\| \, d\mu \to \int_S \|f\| \, d\mu$. But $\left\| \int_S s_n dG \right\| \to \left\| \int_S f dG \right\|$ by (3.4). From the estimation above we know that $\left\| \int_S s_n \, dG \right\| \leq k \int_S \|s_n\| \, d\mu$, for all n. Letting $n \to \infty$ the validity of (3.11) follows. Hence the continuity of the operator $f \to \int_S f dG$ with the $L_1(\mu, X)$ −topology on the space $\mathcal{M}$. Next to prove (b), we shall construct an E−valued integration process on $L_1(\mu, X)$ with the set function G, that coincides with the integral (3.4) on $\mathcal{M}$. This will be the desired extension. Recall that the integral $\int_S s \, dG$, for s simple, has been defined by formula (3.2). Now if $f \in L_1(\mu, X)$, there exist a sequence $s_n \in \mathcal{I}$ such that $\int_S \|f - s_n\| \, d\mu \to 0$. By (3.11) the sequence $\int_S s_n dG$ is fundamental in the Banach space E, so the limit $lim_n \int_S s_n dG$ exists in E and it is easy to check that this limit is independent of the choice of the sequence s_n converging to f in $L_1(\mu, X)$. So we can define:

$$(3.13) \qquad f \in L_1(\mu, X), \qquad \int_S f dG = lim_n \int_S s_n dG$$

where s_n is any sequence in $\mathcal{I}$ converging to f in the $L_1(\mu, X)$ sense.
Now if f is a function in $\mathcal{M}$, every sequence $s_n \in \mathcal{I}$ which converges uniformly to f, converges also in the $L_1(\mu, X)$ sense. So the integrals (3.4) and (3.13) are the same for such f and this proves that (3.13) is an extension of (3.4). To see the inequality (3.12), let $s_n \in \mathcal{I}$ converging in $L_1(\mu, X)$ to the function $f \in L_1(\mu, X)$. By (3.11) we have $\left\| \int_S s_n dG \right\| \leq k \int_S \|s_n\| \, d\mu$, for all n. Taking limits for both sides we get (3.12) from which uniqueness of the extension follows.Part (c) is clear.■

As for the converse of 3.9(c), let us point out the following

3.10 Theorem: *Let $T : L_1(\mu, X) \to E$ be a bounded operator from $L_1(\mu, X)$ to E. Then there exists a unique set function $G : \mathcal{F} \to \mathcal{L}(X, E)$ with finite semivariation satisfying* (3.10), *with the constant $k = \|T\|$ and such that:*

$$(3.14) \qquad f \in L_1(\mu, X), \qquad Tf = \int_S f dG$$

Moreover G is σ−additive in the uniform topology of $\mathcal{L}(X, E)$.

Proof: Define G on $\mathcal{F}$ by the formula:

$$(3.15) \qquad A \in \mathcal{F}, x \in X \qquad G(A).x = T(1_A(\bullet).x)$$

It is clear that $G(A)$ is linear on X for each $A \in \mathcal{F}$ and we have $\|G(A).x\| = \|T(1_A(\bullet).x)\| \leq \|T\| . \mu(A) . \|x\|$. So we deduce that the function G sends $\mathcal{F}$ to $\mathcal{L}(X, E)$ and satisfies $\|G(A)\| \leq \|T\| . \mu(A)$, hence the validity of (3.10) with $k = \|T\|$. On the other hand (3.14) is easily checked from (3.15) for simple functions by linearity, and then extended to arbitrary $f \in L_1(\mu, X)$, by the apropriate limiting process. Finally to get the $\sigma-$additivity of G, let A_n be a sequence in $\mathcal{F}$ with $A_n \searrow \phi$, then $\mu(A_n) \to 0$ and since $\|G(A_n)\| \leq \|T\| .\mu(A_n)$ for all n, we obtain $G(A_n) \to 0$ in the uniform topology of $\mathcal{L}(X, E)$, whence the $\sigma-$additivity of G.■

Now we consider operators T in the class $\mathfrak{D}$. We prove that the operator valued function G attached to an operator $T \in \mathfrak{D}$, according to (3.15), allows an interesting characterization of such operators.

3.16 Theorem: *Let $T : L_1(\mu, X) \to X$ be a bounded operator on $L_1(\mu, X)$ into X. Then T is in the class $\mathfrak{D}$ if and only if the operator valued function attached to it according to* (3.15) *is of the form:*

$$(3.17) \qquad A \in \mathcal{F}, x \in X \qquad G(A).(\bullet) = \lambda(A).I(\bullet)$$

where λ is a bounded measure absolutely continuous with respect to μ, and I is identity operator of X.

Proof: If T is in the class $\mathfrak{D}$, then by the corollary of theorem (2.8) there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$, that is for all $f \in L_1(\mu, X)$, $Tf = \int_S fg \, d\mu$. On the other hand we have from (3.14), $Tf = \int_S fdG$ with G given by (3.15). So taking $f = 1_A(\bullet).x$, for $A \in \mathcal{F}$, $x \in X$, in the two preceding expressions of Tf, we get $G(A).x = \left(\int_A g.x \, d\mu\right) = \left(\int_A g \, d\mu\right).x$. Hence the validity of (3.17) with $\lambda(A) = \int_A g \, d\mu$. Since μ is finite, the function g is in $L_1(\mu)$ and then it is clear that λ is a bounded measure absolutely continuous with respect to μ. Now suppose that the operator valued function attached to T according to (3.15) is of the form: $G(A).x = \lambda(A).x$, with λ a bounded measure absolutely continuous with respect to μ. So we can write $\lambda(A) = \int_A g \, d\mu$, $A \in \mathcal{F}$, for some unique $g \in L_1(\mu)$. Actually the function g belongs to $L_\infty(\mu)$. Indeed by (3.15), $G(A).x = T(1_A(\bullet).x)$ and we deduce that $\|G(A).x\| = \left|\left(\int_A gd\mu\right)\right|.\|x\| \leq \|T\| \mu(A) \|x\|$, which implies $\left|\left(\int_A gd\mu\right)\right| \leq \|T\| \mu(A)$, for all $A \in \mathcal{F}$. Consequently $\|g\|_\infty \leq \|T\|$, that is $g \in L_\infty(\mu)$. Now let us write the formula $G(A).x = \lambda(A).x$ as $\int_S 1_A.x \, dG = \int_S g.1_A.x \, d\mu$, and extend it by linearity to $\int_S s \, dG = \int_S g.s \, d\mu$, for s simple in $L_1(\mu, X)$. If $f \in L_1(\mu, X)$, let s_n be a sequence of simple functions converging to f in $L_1(\mu, X)$. Then $g.s_n$ converges to $g.f$ in $L_1(\mu, X)$, since $g \in L_\infty(\mu)$, so we deduce that $\int_S g.s_n \, d\mu$ goes to $\int_S g.f \, d\mu$. But $\int_S s_n \, dG = \int_S g.s_n \, d\mu$, for all n and by (3.13), $\int_S fdG = lim_n \int_S s_n dG$, consequently $\int_S g.f \, d\mu = \int_S fdG$, for all $f \in L_1(\mu, X)$. But from (3.14) we have, $Tf = \int_S fdG$ for $f \in L_1(\mu, X)$, thus $Tf = \int_S g.f \, d\mu = T_g f$, that is $T \in \mathfrak{D}$.■

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