Jacobic elliptic functions and New Algebraic Method for the systems of Partial Differential Equations

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Abstract

A new algebraic method is devised to obtain a series of exact solutions and more general solutions to a system of Drinfeld- Sokolov-Wilson and New exact solution for a system of nonlinear Euler equations are obtained by using Jacobin elliptic functions based on the idea of the homogeneous balance method [1–5].

1 Introduction

The investigation of the exact solutions of any mathematical model like Nonlinear Euler equation and Drinfeld-Sokolov-wilson play an important role in the study of the physical phenomena. The exact solution, if available, of any physical phenomena facilitates the verification of numerical solvers and aids in the stability method to find some solitary wave solutions of multi-component nonlinear Schrödinger and Klien-Gordan equations. Wang [7] gives the solitary solutions of the approximate equations for long water waves , the coupled KdV equations, dispersive long wave equations in 2+1 dimensions and obtained the solitary solutions of two types of variant Boussinesq equations by using a homogeneous balance method . [2] contract the solutions of the partial differential equations in the hyperbola functions. The idea of homogeneous balance method used to construct the soliton solutions for the nonlinear partial differential equations (for example [6–10]).

Fan [4, 5] use an extended tanh-function method and symbolic computation to obtain the soliton solutions for generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equations and use the Jacobic elliptic functions to contract the soliton on the form $\sum_{i=1}^{m} a_i \mathrm{sn}(x)^i$ solution of system on the elliptic functions.

In this work, we are looking for the soliton solutions of the nonlinear Euler equations of the following type :

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= 2(u_x x + u_y y) \\ v_t + uv_x + vv_y + p_x &= 2(v_x x + v_y y) \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$

We use the Jacobic elliptic functions to contract the solition solutions of nonlinear Euler equations and using a developed a new algebraic method which further exceeds the applicability of the tanh method in obtaining a series of exact solutions of nonlinear equations [11–17]. We apply this method to solve one-dimension Drinfild-Sokolov-Wilson equation

$$\begin{aligned} \psi_t + p\varphi\varphi_x &= 0 \\ \varphi_t + q\varphi_{xxx} + r\psi\varphi_x + s\psi_x\varphi &= 0. \end{aligned} \tag{2}$$

Where p, q, r and s are arbitrary parameters. This is successfully constructing various kinds of exact solutions.

2 $\mathrm{sn}(x) - \mathrm{cn}(x)$ method

For a given a partial differential equation

$$H(u, u_t, u_x, u_y, u_{xx}, \cdots) = 0. \tag{3}$$

We seek its solution in the form

$$u(\xi) = a_0 + \sum_{i=1}^{m+1} \mathrm{sn}^{i-1}(\xi)\,(a_i\mathrm{sn}(\xi) + b_i\mathrm{cn}(\xi)) \tag{4}$$

where $\xi = \lambda_1 t + \lambda_2 x + \lambda_3 y$ and $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ are Jacobic elliptic functions. The parameter m can be found by balancing the highest order linear term with the nonlinear terms. The Jacobi elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ possesses properties of triangular function [1].

$$\begin{aligned} \mathrm{dn}^2(\xi) &= 1 - k^2\mathrm{sn}^2(\xi) \\ (\mathrm{sn}(\xi))' &= \mathrm{sn}(\xi)\mathrm{dn}(xi) \\ (\mathrm{cn}(\xi))' &= -\mathrm{sn}(\xi)\mathrm{dn}(xi) \\ (\mathrm{dn}(\xi))' &= -k^2\mathrm{sn}(\xi)\mathrm{cn}(\xi) \\ \mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) &= 1 \end{aligned} \tag{5}$$

where k is a modules . Substituting (4) into (3) and using (5) will yield a set of algebraic equations, from which can be obtained. In this work, let us consider system (1) by using the method above. We make the transformations $u(t,x,y) = U(\xi)$, $v(t,x,y) = V(\xi)$ and $p(t,x,y) = P(\xi)$ where $\xi = \lambda_1 t + \lambda_2 x + \lambda_3 y$, the system (1) becomes

$$\begin{aligned} U'(\xi)\,[\lambda_1 + \lambda_2 U(\xi) + \lambda_3 V(\xi)] + \lambda_2 P(\xi) - 2(\lambda_2^2 + \lambda_3^2)U''(\xi) &= 0, \\ V'(\xi)\,[\lambda_1 + \lambda_2 U(\xi) + \lambda_3 V(\xi)] + \lambda_3 P(\xi) - 2(\lambda_2^2 + \lambda_3^2)V''(\xi) &= 0, \\ \lambda_2 U'(\xi) + \lambda_3 V'(\xi) &= 0. \end{aligned} \tag{6}$$

Balancing the highest - order linear term with nonlinear term in (6) admits the following ansatz:

$$\begin{aligned}
U(\xi) &= a_0 + a_1\mathrm{sn}(\xi) + a_2\mathrm{cn}(\xi) + a_3\mathrm{sn}^2(\xi) + a_4\mathrm{sn}(\xi)\mathrm{cn}(\xi),\\
V(\xi) &= b_0 + b_1\mathrm{sn}(\xi) + b_2\mathrm{cn}(\xi) + b_3\mathrm{sn}^2(\xi) + b_4\mathrm{sn}(\xi)\mathrm{cn}(\xi),\\
P(\xi) &= ac_0 + c_1\mathrm{sn}(\xi) + c_2\mathrm{cn}(\xi) + c_3\mathrm{sn}^2(\xi) + c_4\mathrm{sn}(\xi)\mathrm{cn}(\xi)
\end{aligned} \tag{7}$$

Substituting (7) into (6) and using maple yield an algebriac system for a_i, b_i, c_i, $(i = 0, 1, 2, 3, 4)$ and $\lambda_1, \lambda_2, \lambda_3$

$$\begin{aligned}
-\lambda_3a_2b_0 - \lambda_2a_2a_0 - \lambda_2c_2 + 2\lambda_3b_2a_3 + 2\lambda_2a_2a_3 + 2\lambda_2a_1a_4 + \lambda_3b_1a_4 - \lambda_1a_2 + \lambda_3a_1b_4 &= 0,\\
-2\lambda_3b_4a_3 - 2\lambda_3b_3a_4 - 4\lambda_2a_4a_3 &= 0,\\
-2\lambda_3b_1a_4 - \lambda_3b_4a_1 - 2\lambda_3b_2a_3 - \lambda_3b_3a_2 - 3\lambda_2a_3a_2 - 3\lambda_2a_4a_1 &= 0,\\
\lambda_1a_4 + \lambda_3b_0a_4 + \lambda_3b_2a_1 + \lambda_2a_2a_1 &= 0,\\
-\lambda_3b_4a_2 + 2\lambda_3b_1a_3 + 3\lambda_2a_1a_3 - 3\lambda_2a_4a_2 - 2\lambda_3a_4b_2 + \lambda_3b_3a_1 &= 0,\\
-2\lambda_2a_4^2 + 2\lambda_3b_3a_3 - 2\lambda_3b_4a_4 + 2\lambda_2a_3^2 &= 0,\\
8\lambda_2^2b_4 + 2\lambda_3^2b_4k^2 + 2\lambda_2^2b_4k^2 + 8\lambda_3^2b_4 &= 0,\\
\lambda_2a_4a_2 + \lambda_3b_2a_4 + \lambda_1a_1 + \lambda_2c_1 + \lambda_3b_0a_1 + \lambda_2a_0a_1 &= 0,\\
\lambda_2c_4 + \lambda_2a_0a_4 &= 0,\\
-2(\lambda_2^2 + \lambda_3^2)b_4 &= 0,\\
-2(\lambda_2^2 + \lambda_3^2)b_2 &= 0,\\
-2(\lambda_2^2 + \lambda_3^2)b_0 &= 0,\\
-2(\lambda_2^2 + \lambda_3^2)b_1 &= 0,\\
-2(\lambda_2^2 + \lambda_3^2)b_3 &= 0,\\
2\lambda_2a_3 + 2\lambda_3b_3 &= 0,\\
\lambda_2a_1 + \lambda_3b_1 &= 0,\\
-\lambda_2a_2 - \lambda_3b_2 &= 0,\\
\lambda_2a_4 + \lambda_3b_4 &= 0,\\
-2\lambda_2b_4a_3 - 4\lambda_3b_3b_4 - 2\lambda_2a_4b_3 &= 0,\\
\lambda_2b_1a_3 + 3\lambda_3b_3b_1 - \lambda_2a_4b_2 - 3\lambda_3b_4b_2 - 2\lambda_2a_2a_4 + 2\lambda_2a_1b_3 &= 0,\\
-2_3b_3^2 - 2\lambda_2b_4a_4 + 2\lambda_2b_3a_3 - 2\lambda_3b_4^2 &= 0,\\
\lambda_2b_4a_2 + \lambda_1b_1 + \lambda_3c_1 + \lambda_2b_1a_0 + \lambda_3b_2b_4 + \lambda_3b_1b_0 &= 0\\
8\lambda_3^2b_3k^2 + 8\lambda_3^2b_3 + 8\lambda_2^2b_3k^2 + 8\lambda_2^2b_3 &= 0,\\
2\lambda_3^2b_1 + 2\lambda_3^2b_1 + 2\lambda_3^2b_1k^2 + 2\lambda_2^2b_1k^2 &= 0,\\
\lambda_2b_4a_0 + \lambda_3c_4 + \lambda_3b_4b_0 + \lambda_3b_1b_2 + \lambda_2a_2b_1 + \lambda_1b_4 &= 0,\\
-\lambda_2a_3b_2 - 3\lambda_3b_2b_3 - \lambda_2a_4b_1 - 3\lambda_3b_1b_4 - 2\lambda_2a_1b_4 - 2\lambda_2b_3a_2 &= 0,\\
-2\lambda_1b_3 - \lambda_3b_2^2 - \lambda_2b_2a_2 + \lambda_2b_4a_4 + \lambda_2b_1a_1 + 2\lambda_3c_3 + \lambda_3b_4^2 - \lambda_3b_1^2 + 2\lambda_2b_0b_3 + 2\lambda_2b_3a_0 &= 0,\\
-\lambda_1b_2 - \lambda_2b_2a_0 + 2\lambda_3b_2b_3 - \lambda_3b_0b_2 + \lambda_2b_1a_4 + \lambda_2a_1b_4 + 2\lambda_3b_1b_4 - \lambda_3c_2 + 2\lambda_2a_2b_3 &= 0,\\
2\lambda_2a_4b_3 - (\lambda_2a_2 - 2\lambda_3b_1)b_1 - 2\lambda_3c_4 - \lambda_2a_1b_2 + (\lambda_2a_3 - 2\lambda_1 - 2\lambda_2a_0 + 3\lambda_3b_3 - 2\lambda_3b_0)b_4 &= 0,\\
-(\lambda_3b_1 + 2\lambda_2a_1)a_2 - \lambda_3b_2a_1 - (2\lambda_3b_0 + 2\lambda_2a_0 - \lambda_3b_3 - 6\lambda_1\lambda_2a_3)a_4 - 2\lambda_2c_4 + 2\lambda_3a_3b_4 &= 0,\\
2\lambda_3a_3b_0 - a_2^2\lambda_2 - \lambda_3b_2a_2 + 2\lambda_1a_3 + 2\lambda_2a_0a_3 + \lambda_3b_1a_1 + 2\lambda_2c_3 + a_4^2\lambda_2 + a_1^2\lambda_2 + \lambda_3b_4a_4 &= 0.
\end{aligned}$$

With the aid of Maple, we find the set of solutions:

Case 1.

$$a_0 = \frac{ic_3}{b_3}, \qquad a_4 = ib_4, \qquad \lambda_1 = -\lambda_3 b_0,$$
$$a_2 = ib_2, \qquad c_2 = \frac{c_3 b_2}{b_3}, \qquad \lambda_2 = i\lambda_3 \tag{8}$$
$$a_3 = ib, \qquad c_4 = \frac{c_3 b_4}{b_3}, \qquad a_1 = b_1 = c_1 = 0$$

where $b_0, b_2, b_3, b_4, \lambda_3, c_0$ and c_3 are arbitrary constants.

Case 2.

$$a_0 = \frac{i}{b_1\lambda_3}(\lambda_1 b_1 + \lambda_3 c_1 + \lambda_3 b_0 b_1), \qquad a_1 = ib_1,$$
$$a_2 = a_3 = a_4 = b_2 = b_3 = b_4 = c_2 = c_3 = c_4 = 0 \qquad \lambda_2 = i\lambda_3, \tag{9}$$

where $b_0, b_1, \lambda_1, \lambda_3, c_1$ and c_0 are arbitrary constants.

Case 3.

$$a_1 = ib_1, \qquad \lambda_2 = i\lambda_3,$$
$$c_1 = \frac{b_1 c_2}{b_2}, \qquad a_0 = \frac{i}{b_2\lambda_3}(\lambda_1 b_2 + \lambda_3 c_2 + \lambda_3 b_0 b_2), \tag{10}$$
$$a_2 = ib_2 \qquad b_3 = b_4 = a_3 = a_4 = c_3 = c_4 = 0$$

where $b_0, b_1, b_2, \lambda_1, \lambda_3, c_2$ and c_0 are arbitrary constants.

Case 4.

$$a_1 = ib_1, \qquad \lambda_1 = -i\lambda_3 b_0, \qquad a_2 = ib_2,$$
$$c_1 = \frac{b_1 c_2}{b_2}, \qquad \lambda_2 = i\lambda_3, \qquad a_4 = ib_4 \tag{11}$$
$$c_4 = \frac{b_4 c_2}{b_2}, \qquad a_0 = \frac{ic_2}{b_2}, \qquad b_3 = a_3 = c_3 = 0$$

where $b_0, b_1, b_2, b_4, \lambda_3, c_2$ and c_0 are arbitrary constants.

Case 5.

$$a_1 = ib_1, \qquad \lambda_2 = i\lambda_3, \qquad a_2 = ib_2,$$
$$a_3 = ib_3, \qquad c_1 = \frac{c_3 b_1}{b_3}, \qquad c_2 = \frac{b_2 c_3}{b_3}$$
$$a_0 = \frac{i}{b_3\lambda_3}(\lambda_1 b_3 + \lambda_3 c_+ 3 + \lambda_3 b_0 b_3) \tag{12}$$

where $b_0, b_1, b_2, b_3, \lambda_1, \lambda_3, c_3$ and c_0 are arbitrary constants.

Case 6.

$$a_1 = ib_1, \qquad \lambda_1 = -i\lambda_3 b_0, \qquad a_4 = ib_4,$$
$$c_1 = \frac{b_1 c_3}{b_3}, \qquad a_3 = ib_3, \qquad \lambda_2 = i\lambda_3 \tag{13}$$
$$a_0 = \frac{ic_3}{b_3}, \qquad a_2 = b_2 = c_2 = 0,$$

where $b_0, b_1, b_3, b_4, \lambda_3, c_3$ and c_0 are arbitrary constants.

Case 7.

$$
\begin{array}{lll}
c_2 = -ib_2a_0, & c_3 = -ib_3a_0, & c_4 = -ib_4a_0, \\
\lambda_1 = -\lambda_3 b_0, & a_1 = ib_1, & c_1 = -ia_0b_1 \\
\lambda_2 = i\lambda_3, & a_2 = ib_2, & a_4 = ib_4 \\
a_3 = ib_3 & &
\end{array} \qquad (14)
$$

where $b_0, b_1, b_2, b_3, b_4, \lambda_3, a_0$ and c_0 are arbitrary constants.

3 Class of Solution

In this way, we able to construct the wave solution for the system (1) in the form

Case 1. the solution of the system (1) with (8) takes the form

$$
\begin{aligned}
U(\xi) &= \frac{ic_3}{b_3} + ib_2\text{cn}(\xi) + ib_3\text{sn}^2(\xi) + ib_4\text{cn}(\xi)\text{sn}(\xi) \\
V(\xi) &= b_0 + b_2\text{cn}(\xi) + b_3\text{sn}^2(\xi) + b_4\text{cn}(\xi)\text{sn}(\xi) \\
P(\xi) &= c_0 + \frac{b_2c_3}{b_3}\text{cn}(\xi) + c_3\text{sn}^2(\xi) + \frac{c_3b_4}{b_3}\text{cn}(\xi)\text{sn}(\xi)
\end{aligned}
$$

where $b_0, b_2, b_3, b_4, \lambda_3, c_0$ and c_4 are arbitrary constants and $\xi = -\lambda_3 b_0 t + \lambda_3(ix + y)$.

Case 2. the solution of the system (1) with (9) takes the form

$$
\begin{aligned}
U(\xi) &= \frac{i}{b_1\lambda_3}(\lambda_1 b_1 + \lambda_3 c_1 + \lambda_3 b_0 b_1) + ib_1\text{sn}(\xi) \\
V(\xi) &= b_0 + b_1\text{sn}(\xi) \\
P(\xi) &= c_0 + c_1\text{sn}(\xi)
\end{aligned}
$$

where b_0, b_1, λ_3, c_0 and c_1 are arbitrary constants and $\xi = \lambda_1 t + \lambda_3(ix + y)$.

Case 3. the solution of the system (1) with (10) takes the form

$$
\begin{aligned}
U(\xi) &= \frac{i}{b_2\lambda_3}(\lambda_1 b_2 + \lambda_3 c_2 + \lambda_3 b_0 b_2) + ib_1\text{sn}(\xi) + ib_2\text{cn}(\xi) \\
V(\xi) &= b_0 + b_1\text{sn}(\xi) + b_2\text{cn}(\xi) \\
P(\xi) &= c_0 + \frac{b_1c_2}{b_2}\text{sn}(\xi) + c_2\text{cn}9\xi)
\end{aligned}
$$

where $b_0, b_1, b_2, \lambda_3, \lambda_1, c_0$ and c_2 are arbitrary constants and $\xi = \lambda_1 t + \lambda_3(ix + y)$.

Case 4. the solution of the system (1) with (11) takes the form

$$
\begin{aligned}
U(\xi) &= \frac{ic_2}{b_2} + ib_1\text{sn}(\xi) + ib_2\text{cn}(\xi) + ib_4\text{cn}(\xi)\text{sn}(\xi) \\
V(\xi) &= b_0 + b_1\text{sn}(\xi) + b_2\text{cn}(\xi) + b_4\text{cn}(\xi)\text{sn}(\xi) \\
P(\xi) &= c_0 + \frac{b_1c_2}{b_2}\text{sn}(\xi) + c_2\text{cn}(\xi) + \frac{c_2b_4}{b_2}\text{cn}(\xi)\text{sn}(\xi)
\end{aligned}
$$

where $b_0, b_1, b_2, b_4, \lambda_3, c_0$ and c_2 are arbitrary constants and $\xi = -\lambda_3 b_0 t + \lambda_3(ix + y)$.

Case 5. the solution of the system (1) with (12) takes the form

$$U(\xi) = \frac{i}{b_3\lambda_3}(\lambda_1 b_3 + \lambda_3 c_+ 3 + \lambda_3 b_0 b_3) + ib_1\mathrm{sn}(\xi) + ib_2\mathrm{cn}(\xi) + ib_3\mathrm{sn}^2(\xi)$$
$$V(\xi) = b_0 + b_1\mathrm{sn}(\xi) + b_2\mathrm{cn}(\xi) + b_3\mathrm{sn}^2(\xi)$$
$$P(\xi) = c_0 + \frac{b_1 c_3}{b_3} + \frac{b_2 c_3}{b_3}\mathrm{cn}(\xi) + c_3\mathrm{sn}^2(\xi)$$

where $b_0, b_1, b_2, b_3, \lambda_1, \lambda_3, c_0$ and c_3 are arbitrary constants and $\xi = -\lambda_1 t + \lambda_3(ix + y)$.

Case 6. the solution of the system (1) with (13) takes the form

$$U(\xi) = \frac{ic_3}{b_3} + ib_1\mathrm{sn}(\xi) + a_3\mathrm{sn}^2(\xi) + ib_4\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$
$$V(\xi) = b_0 + b_1\mathrm{sn}(\xi) + b_3\mathrm{sn}^2(\xi) + b_4\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$
$$P(\xi) = c_0 + \frac{b_1 c_3}{b_3}\mathrm{sn}(\xi) + c_3\mathrm{sn}^2(\xi) + \frac{c_3 b_4}{b_3}\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$

where $b_0, b_1, b_3, b_4, \lambda_3, c_0$ and c_3 are arbitrary constants and $\xi = -\lambda_3 b_0 t + \lambda_3(ix + y)$.

Case 7. the solution of the system (1) with (14) takes the form

$$U(\xi) = a_0 + ib_1\mathrm{sn}(\xi) + ib_2\mathrm{cn}(\xi) + ib_3\mathrm{sn}^2(\xi) + ib_4\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$
$$V(\xi) = b_0 + b_1\mathrm{sn}(\xi) + b_2\mathrm{cn}(\xi) + b_3\mathrm{sn}^2(\xi) + b_4\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$
$$P(\xi) = c_0 - ia_0 b_1\mathrm{sn}(\xi) - ia_0 b_2\mathrm{cn}(\xi) - ib_3 a_0\mathrm{sn}^2(\xi) - ia_0 b_4\mathrm{cn}(\xi)\mathrm{sn}(\xi)$$

where $b_0, b_1, b_2, b_3, b_4, \lambda_3, c_0$ and a_0 are arbitrary constants and $\xi = -\lambda_3 b_0 t + \lambda_3(ix + y)$.

4 An Algebraic Method

For Drinfeld-Soklov-Wilson equations

$$\begin{aligned} \psi_t + p\varphi\varphi_x &= 0, \\ \varphi_t + q\varphi_{xx} + r\psi\varphi_x + s\varphi\psi_x &= 0, \end{aligned} \tag{15}$$

where p, q, r and s are arbitrary parameters. Likewise, let $\psi(t,x) = \psi^*(\xi)$, $\varphi(t,x) = \varphi^*(\xi)$ and $\xi = x - ct$, then the system (1) becomes

$$\begin{aligned} c\psi^{*\prime} - p\varphi^*\varphi^{*\prime} &= 0, \\ c\varphi^{*\prime} - q\varphi^{*\prime\prime\prime} - r\psi^*\varphi^{*\prime} - s\varphi^*\psi^{*\prime} &= 0. \end{aligned} \tag{16}$$

From balancing idea we get

$$\begin{aligned} \psi^* &= b_0 + b_1\hat{\varphi} + b_2\hat{\varphi}^2(\xi), \\ \varphi^* &= a_0 + a_1\hat{\varphi}(\xi), \end{aligned} \tag{17}$$

where

$$\hat{\varphi}'(\xi) = m\sqrt{c_0 + c_1\hat{\varphi}(\xi) + c_2\hat{\varphi}^2(\xi) + c_3\hat{\varphi}^3(\xi) + c_4\hat{\varphi}^4(\xi)}.$$

Substituting (17) into (16) and using Maple, we obtain a system of algebraic equations

$$\begin{aligned} 2b_2mc - a_1^2mp &= 0, \\ b_1cm - pma_1a_0 &= 0, \\ -3qa_1m^3c_3 - rma_1b_1 - sma_1b_1 - 2sma_0b_2 &= 0, \\ -rma_1b_2 - 6qma_1c_4 - 2sma_1b_2 &= 0, \\ cma_1 - qm^3a_1c_2 - sma_0b_1 - rma_1b_0 &= 0. \end{aligned} \tag{18}$$

Solving system (18), we get a solution of the form

$$\begin{aligned} b_2 &= \frac{1}{4}pa_1^2, & b_1 &= \frac{pa_1a_0}{c}, \\ b_0 &= \frac{-c - qm^2c_2 - spa_0^2/c}{r}, & c_3 &= -\frac{pa_1a_0(r+2s)}{3qm^2c}, \\ c_4 &= -\frac{pa_1^2(r+2s)}{12qm^2c}, \end{aligned} \tag{19}$$

with a_0, a_1, c_0, c_1, c_2 are arbitrary. Take $a_0 = 0$ the system (19) becomes

$$\begin{aligned} b_2 &= -\frac{1}{2}pa_1^2, & b_1 &= 0, \\ b_0 &= \frac{-c - qm^2c_2}{r}, & c_3 &= 0, \\ c_4 &= \frac{pa_1^2(r+2s)}{12qm^2}, \end{aligned} \tag{20}$$

with a_1, c_0, c_1, c_2 are arbitrary.

I. When $c_0 = c_1 = c_3 = 0$, we obtained

$$\hat{\varphi} = \sqrt{\frac{-c_2}{c_4}}\operatorname{sech}(-\sqrt{c_2}\xi)$$

, Then the solution in (17) take the form

1. When $c_2 \succ 0$, $c_4 \prec 0$ then

$$\psi^* = \frac{-c - qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[\sqrt{\frac{-12qm^2c_2}{pa_1^2(r+2s)}}\operatorname{sech}\sqrt{c_2}\xi\right]^2$$

$$\varphi^* = a_1\sqrt{\frac{-12qm^2c_2}{pa_1^2(r+2s)}}\operatorname{sech}\sqrt{c_2}\xi.$$

2. When $c_2 \prec 0$ and one of p, q or r is negative and the rest is nonnegative with $s > 0$ then we have

$$\psi^* = \frac{-c - qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[\sqrt{\frac{-12qm^2c_2}{pa_1^2(r+2s)}}\operatorname{sech}\left[-\sqrt{-c_2}\xi\right]\right]^2$$

$$\varphi^* = a_1\sqrt{\frac{-12qm^2c_2}{pa_1^2(r+2s)}}\operatorname{sech}\left[-\sqrt{-c_2}\xi\right].$$

II. When $c_2 \succ 0$ and $c_0 = \dfrac{12c_2^2 m^{*2}(1-m^{*2})qm^2}{pa_1^2(r+2s)(2m^{*2}-1)^2}$, then we have

$$\hat{\varphi} = \sqrt{\frac{-c_0}{c_2(1-m^{*2})}}\,\mathrm{cn}\sqrt{\frac{c_2}{(2m^{*2}-1)}}\xi$$

and hence we have

$$\psi^* = \frac{-c-qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[\sqrt{\frac{-m^{*2}c_2}{c_4(2m^{*2}-1)^2}}\,\mathrm{cn}\sqrt{\frac{c_2}{(2m^{*2}-1)}}\xi\right]^2$$

$$\varphi^* = a_1\sqrt{\frac{-m^{*2}c_2}{c_4(2m^{*2}-1)}}\,\mathrm{cn}\sqrt{\frac{c_2}{(2m^{*2}-1)}}\xi.$$

Where m^* is modulus will appear in the Jacobic doubly periodic solutions.

III. When $c_2 \prec 0$ and $c_0 = \dfrac{12c_2^2 m^{*2}qm^2}{pa_1^2(r+2s)(m^{*2}+1)}$, in this case the soliton solution takes the form

$$\psi^* = \frac{-c-qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[m\sqrt{\frac{-12c_2m^{*2}qm^2}{pa_1^2(r+2s)(m^{*2}+1)}}\,\mathrm{sn}\sqrt{\frac{-c_2}{(m^{*2}+1)}}\xi\right]^2$$

$$\varphi^* = a_1m\sqrt{\frac{-12c_2m^{*2}qm^2}{pa_1^2(r+2s)(m^{*2}+1)}}\,\mathrm{sn}\sqrt{\frac{-c_2}{(m^{*2}+1)}}\xi.$$

IV. If $c_3 = c_1 = 0$ and $c_0 = \dfrac{12c_2^2qm^2}{pa_1^2(r+2s)}$ where c_2 and a_1 are arbitrary, $c_4 = \dfrac{pa_1^2(r+2s)}{12qm^2}$, consequently, the soliton solution takes the form depending on the parameters

(1) If $c_2 \prec 0$ ($p \prec 0,\ q,r,s \succ 0$ or $q \prec 0,\ p,r,s \succ 0$). In this case if $m^* \longrightarrow 1$ then $\mathrm{sn}\xi \longrightarrow \tanh\xi$, consequently

$$\psi^* = \frac{-c-qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[m\sqrt{\frac{-12c_2qm^2}{pa_1^2(r+2s)}}\tanh\sqrt{\frac{-c_2}{2}}\xi\right]^2$$

$$\varphi^* = a_1m\sqrt{\frac{-12c_2qm^2}{pa_1^2(r+2s)}}\tanh\sqrt{\frac{-c_2}{2}}\xi.$$

(2) If $c_2 \succ 0$, and $p,q,r,s \succ 0$, and $m^* \longrightarrow 0$

$$\psi^* = \frac{-c-qm^2c_2}{r} - \frac{1}{2}pa_1^2\left[m\sqrt{\frac{-12c_2qm^2}{pa_1^2(r+2s)}}\tan\sqrt{\frac{-c_2}{2}}\xi\right]^2$$

$$\varphi^* = a_1m\sqrt{\frac{-12c_2qm^2}{pa_1^2(r+2s)}}\tan\sqrt{\frac{-c_2}{2}}\xi.$$

V. When $r = -2s$, then $c_4 = 0$ and if $c_3 = 0$, c_0, c_1 and c_2 are arbitrary constants, then we obtained

$$\hat{\varphi} = \sqrt{\frac{c_0}{c_1}} \sinh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}.$$

In this case the soliton solution take the form

(1) If $\frac{c_0}{c_2} \succ 0$, then

$$\psi^* = \frac{-c - qm^2 c_2 + spa_0^2}{r} - pa_1 a_0 \left[\sqrt{\frac{c_0}{c_1}} \sinh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}\right] + \frac{1}{2} pa_1^2 \left[\sqrt{\frac{c_0}{c_1}} \sinh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}\right]^2$$

$$\varphi^* = a_0 + a_1 \sqrt{\frac{c_0}{c_1}} \sinh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}.$$

(2) If $\frac{c_0}{c_2} \prec 0$, then

$$\psi^* = \frac{-c - qm^2 c_2 + spa_0^2}{r} - pa_1 a_0 \left[\sqrt{\frac{c_0}{c_1}} \cosh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}\right] + \frac{1}{2} pa_1^2 \left[\sqrt{\frac{c_0}{c_1}} \cosh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}\right]^2$$

$$\varphi^* = a_0 + a_1 \sqrt{\frac{c_0}{c_1}} \cosh(\sqrt{c_0} m\xi) + \frac{c_2}{c_1}.$$

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