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Left Hopf Algebras and Self Duality

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Abstract

In this paper, we introduce the concept of a left bicrossproduct Hopf algebra associated to a factorization of a finite group X into a subgroup G and a subsemigroup M. Moreover, we show that for a left Hopf algebra $H = kM \bowtie k(G)$ associated to a factorization $X = GM$ of a group X into a subgroup G and a subsemigroup M with identity and left inverse property, there is a left Hopf algebra isomorphism $H \to H^*$ which sends basis elements to basis elements can be constructed from a factor-reversing isomorphism of $X = GM$ and vise versa.

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1 Introduction

Bicrossproducts which are associated to a factorization of groups are essential in the field of non-commutative and non-cocommutative Hopf algebras. Bicrossproduct Hopf algebras have many applications in quantum mechanics and geometry and the interrelation between them (see [8]). These algebras and their dual were extensively studied in [1], [2], [4], [5] and [8].

In [4], Beggs, Gould and Majid showed that basis-preserving self-duality structures for the bicrossproduct Hopf algebras are in one-to-one correspondence with factor-reversing group isomorphisms.

In [6], Green, Nichols and Taft defined a left Hopf algebra to be a k-bialgebra $(B, m, \Delta, \mu, \epsilon : k)$ with a left antipode S, i.e., $S \in Hom_k(B, B)$ and $S * id = \mu\epsilon$.

In This paper, we generalize some results of [4] using this definition of left Hopf algebra in specific case. More specifically, we show that for a left

Hopf algebra $H = kM \bowtie k(G)$ associated to the factorization $X = GM$ of a group X into a subgroup G and a subsemigroup M with identity and left inverse property, where $k(G)$ is the Hopf algebra of function on G and kM is the semigroup left Hopf algebra of M, there is a left Hopf algebra isomorphism $H \to H^*$ which sends basis elements to basis elements can be constructed from a factor-reversing isomorphism of $X = GM$. Conversely, we show that for a factorization $X = GM$ of a group X into a subgroup G and a subsemigroup M whit identity and a left inverse property, a factor-reversing semigroup isomorphism of $X = GM$ can be obtained from a left Hopf algebra self-duality pairings $<,>: H \otimes H \to k$ on the left Hopf algebra $H = kM \bowtie k(G)$.

2 Self-duality of bicrossproducts

Here we introduce the concept of bicrossproduct left Hopf algebras associated to factorization of a group into a subgroup and a subsemigroup with identity and a left inverse property. The left inverse for an element $m \in M$ will be denoted by m^L, if it exists. We need the following definitions:

Definition 2.1 *let $X = GM$ be a factorization of a group into a subgroup G and a subsemigroup with identity and a left inverse property M. A bialgebra $H = kM \bowtie k(G)$ with basis $m \otimes \delta_g$ where m in a subsemigroup M and g in a subgroup G is called a left Hopf algebra if there is a one-sided antipode map S such that*

$$S(m \otimes \delta_g) = (m \triangleleft g)^L \otimes \delta_{(m \triangleright g)^{-1}}.$$

Definition 2.2 *Let $X = GM$ be a group factorization. We define a semigroup isomorphism $\theta : X \to X$ to be factor-reversing if $\theta(G) \subset M$ and $\theta(M) \subset G$.*

Now, let $X = GM$ be a group which factorizes into a subgroup G and a subsemigroup with identity M. Then M acts on G through the right action $\triangleright: M \times G \to G$ and G acts on M through the left action $\triangleleft: M \times G \to M$. These actions are defined by the unique factorization

$$mg = (m \triangleright g)(m \triangleleft g), \tag{1}$$

where $m \in M$ and $g \in G$. According to [4], it is easy to show that these actions obeying the following conditions for all $m, m_1 \in M$ and $g, g_1 \in G$:

$$m \triangleleft e = m, (m \triangleleft g) \triangleleft g_1 = m \triangleleft (gg_1); e \triangleleft g = e, \tag{2}$$

$$(mm_1) \triangleleft g = (m \triangleleft (m_1 \triangleright g))(m_1 \triangleleft g), \tag{3}$$

$$e \triangleright g = g, m \triangleright (m_1 \triangleright g) = (mm_1) \triangleright g; m \triangleright e = e, \tag{4}$$

$$m \triangleright (gg_1) = (m \triangleright g)((m \triangleleft g) \triangleright g_1). \tag{5}$$

It can be seen that we can associate to this factorization a bicrossproduct bialgebra $H = kM \bowtie k(G)$ with basis $m \otimes \delta_g$ where $m \in M$ and $g \in G$. The product, unit, coproduct and counit are defined as follows:

$$(m \otimes \delta_g)(m_1 \otimes \delta_{g_1}) = \delta_{g,m_1 \triangleright g_1}(mm_1 \otimes \delta_{g_1}), \tag{6}$$

$$1_H = \sum_g e \otimes \delta_g\,, \tag{7}$$

$$\Delta(m \otimes \delta_g) = \sum_{x,y \in G: xy=g} m \otimes \delta_x \otimes (m \triangleleft x) \otimes \delta_y\,, \tag{8}$$

$$\epsilon_H(m \otimes \delta_g) = \delta_{g,e}\,. \tag{9}$$

If M posses a left inverse m^L for each $m \in M$, then H becomes a left Hopf algebra and the antipode will be given by:

$$S(m \otimes \delta_g) = (m \triangleleft g)^L \otimes \delta_{(m \triangleright g)^{-1}}. \tag{10}$$

Due to these formulas, it can be noted that $H = kM \bowtie k(G)$ has the smash product algebra structure by the induced action of M and the smash coproduct coalgebra structure by the induced coaction of G.

In the symbol $H = kM \bowtie k(G)$, kM is the semigroup left Hopf algebra of the semigroup with identity and the left inverse property M. A basis of kM is given by the elements of M, with multiplication given by the semigroup product in M, and comultiplication given by $\Delta m = m \otimes m$ for $m \in M$. Also, $k(G)$ is the Hopf algebra of functions on G with basis given by δ_g for $g \in G$. The product is just multiplication of functions, and the coproduct is

$$\Delta\delta_g = \sum_{x,y \in G: xy=g} \delta_x \otimes \delta_y.$$

In addition, a dual bicrossproduct bialgebra $H^* = k(M) \blacktriangleright\!\!\triangleleft\, kG$ can be defined with basis $\delta_m \otimes g$ where $m \in M$ and $g \in G$. The product, unit, coproduct and counit are defined as follows:

$$(\delta_m \otimes g)(\delta_{m_1} \otimes g_1) = \delta_{m \triangleleft g, m_1}(\delta_m \otimes gg_1), \tag{11}$$

$$1_{H^*} = \sum_m \delta_m \otimes e\,, \tag{12}$$

$$\Delta(\delta_m \otimes g) = \sum_{a,b \in M : ab = m} \delta_a \otimes (b \triangleright g) \otimes \delta_b \otimes g\,, \tag{13}$$

$$\epsilon_{H^*}(\delta_m \otimes g) = \delta_{m,e}\,. \tag{14}$$

If M posses a left inverse m^L for each $m \in M$, then H^* becomes a left Hopf algebra and the antipode will be given by:

$$S(\delta_m \otimes g) = \delta_{(m \triangleleft g)^L} \otimes (m \triangleright g)^{-1}\,. \tag{15}$$

Proposition 2.3 *For a left Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ associated to a factorization $X = GM$ of a group X into a subgroup G and a subsemigroup M with identity and left inverse property, where $k(G)$ is the Hopf algebra of function on G and kM is the semigroup left Hopf algebra of M, there is a left Hopf algebra isomorphism $H \to H^*$, which sends basis elements to basis elements, can be constructed from a factor-reversing isomorphism of $X = GM$.*

Proof. We define a linear map $\widetilde{\theta} : H \to H^*$ by

$$\widetilde{\theta}(m \otimes \delta_g) = \delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g) \tag{16}$$

and verify that this is a left Hopf algebra isomorphism $\widetilde{\theta} : kM \triangleright\!\!\blacktriangleleft k(G) \to k(M) \blacktriangleright\!\!\triangleleft\, kG$ whenever θ is a semigroup isomorphism. Suppose that θ is a semigroup isomorphism. Then

$$\begin{aligned}\theta(m_1 g_1) &= \theta((m_1 \triangleright g_1)(m_1 \triangleleft g_1)) \\ &= \theta(m_1 \triangleright g_1)\theta(m_1 \triangleleft g_1),\end{aligned}$$

and

$$\theta(m_1 g_1) = \theta(m_1)\theta(g_1).$$

The condition that these two expressions are the same is that, for all m_1 and g_1,

$$\begin{aligned}\theta(m_1)\theta(g_1) &= \theta(m_1 \triangleright g_1)\theta(m_1 \triangleleft g_1)\\ &= (\theta(m_1 \triangleright g_1) \triangleright \theta(m_1 \triangleleft g_1))(\theta(m_1 \triangleright g_1) \triangleleft \theta(m_1 \triangleleft g_1)).\end{aligned}$$

So, by the uniqueness of factorization, we get

$$\theta(m_1) = \theta(m_1 \triangleright g_1) \triangleright \theta(m_1 \triangleleft g_1), \tag{17}$$

and

$$\theta(g_1) = \theta(m_1 \triangleright g_1) \triangleleft \theta(m_1 \triangleleft g_1). \tag{18}$$

Now, to prove that $\widetilde{\theta}$ is a left Hopf algebra isomorphism, we check the conditions for $\widetilde{\theta}$ to be an algebra isomorphism, i.e.,

$$\widetilde{\theta}((m \otimes \delta_g)(m_1 \otimes \delta_{g_1})) = \widetilde{\theta}(m \otimes \delta_g)\widetilde{\theta}(m_1 \otimes \delta_{g_1}),$$

which we do as follows:

$$\begin{aligned}\widetilde{\theta}((m \otimes \delta_g)(m_1 \otimes \delta_{g_1})) &= \widetilde{\theta}(\delta_{g,m_1 \triangleright g_1}(mm_1 \otimes \delta_{g_1}))\\ &= \delta_{g,m_1 \triangleright g_1}\delta_{\theta(mm_1 \triangleright g_1)} \otimes \theta(mm_1 \triangleleft g_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{\theta}(m \otimes \delta_g)\widetilde{\theta}(m_1 \otimes \delta_{g_1}) &= (\delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g))(\delta_{\theta(m_1 \triangleright g_1)} \otimes \theta(m_1 \triangleleft g_1))\\ &= \delta_{\theta(m \triangleright g) \triangleleft \theta(m \triangleleft g), \theta(m_1 \triangleright g_1)}(\delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g)\theta(m_1 \triangleleft g_1))\\ &= \delta_{\theta(g),\theta(m_1 \triangleright g_1)}\delta_{\theta(m \triangleright g)} \otimes \theta((m \triangleleft g)(m_1 \triangleleft g_1)) \quad \text{(applying } \theta^{-1}\text{)}\\ &= \delta_{g,m_1 \triangleright g_1}\delta_{m \triangleright g} \otimes \theta((m \triangleleft g)(m_1 \triangleleft g_1)) \qquad \text{(putting g = m}_1 \triangleright \text{g}_1\text{)}\\ &= \delta_{g,m_1 \triangleright g_1}\delta_{\theta(m \triangleright (m_1 \triangleright g_1))} \otimes \theta((m \triangleleft (m_1 \triangleright g_1))(m_1 \triangleleft g_1))\\ &= \delta_{g,m_1 \triangleright g_1}\delta_{\theta(mm_1 \triangleright g_1)} \otimes \theta(mm_1 \triangleleft g_1).\end{aligned}$$

Next, we check the condition for $\widetilde{\theta}$ to be a coalgebra isomorphism, i.e.,

$$\Delta\widetilde{\theta}(m \otimes \delta_g) = (\widetilde{\theta} \otimes \widetilde{\theta})\Delta(m \otimes \delta_g). \tag{19}$$

We start with

$$\begin{aligned}\Delta\widetilde{\theta}(m \otimes \delta_g) &= \Delta(\delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g))\\ &= \sum_{a,b \in M : ab = \theta(m \triangleright g)} \delta_a \otimes (b \triangleright \theta(m \triangleleft g)) \otimes \delta_b \otimes \theta(m \triangleleft g).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\widetilde{\theta}\otimes\widetilde{\theta})\Delta(m\otimes\delta_g) &= (\widetilde{\theta}\otimes\widetilde{\theta})\sum_{x,y\in G:xy=g}\big(m\otimes\delta_x\big)\otimes\big((m\lhd x)\otimes\delta_y\big)\\ &= \sum_{x,y\in G:xy=g}\widetilde{\theta}(m\otimes\delta_x)\otimes\widetilde{\theta}((m\lhd x)\otimes\delta_y)\\ &= \sum_{x,y\in G:xy=g}\delta_{\theta(m\rhd x)}\otimes\theta(m\lhd x)\otimes\delta_{\theta((m\lhd x)\rhd y)}\otimes\theta((m\lhd x)\lhd y)\\ &= \sum_{x,y\in G:xy=g}\delta_{\theta(m\rhd x)}\otimes\theta(m\lhd x)\otimes\delta_{\theta((m\lhd x)\rhd y)}\otimes\theta(m\lhd xy).\end{aligned}$$

If we put $a=\theta(m\rhd x)$ and $b=\theta((m\lhd x)\rhd y)$, then

$$\begin{aligned}ab &= \theta(m\rhd x)\theta((m\lhd x)\rhd y)\\ &= \theta((m\rhd x)((m\lhd x)\rhd y))\\ &= \theta(m\rhd (xy))=\theta(m\rhd g).\end{aligned}$$

By comparing the left hand side of equation (19) with the right hand side, we should have

$$\begin{aligned}b\rhd\theta(m\lhd g) &= \theta((m\lhd x)\rhd y)\rhd\theta(m\lhd g)\\ &= \theta((m\rhd x)^{-1}(m\rhd (xy)))\rhd\theta(m\lhd g)\\ &= \theta((m\rhd x)^{-1}(m\rhd g))\rhd\theta(m\lhd g)\\ &= \theta(m\rhd x)^{-1}\rhd(\theta(m\rhd g)\rhd\theta(m\lhd g))\\ &= \theta(m\rhd x)^{-1}\rhd\theta(m)\\ &= \theta(m\rhd x)^{-1}\rhd(\theta(m\rhd x)\rhd\theta(m\lhd x))\\ &= \theta((m\rhd x)^{-1}(m\rhd x))\theta(m\lhd x)\\ &= \theta(e)\theta(m\lhd x)\\ &= \theta(m\lhd x).\end{aligned}$$

This shows that equation (19) is satisfied.

We need now to check the effect of $\widetilde{\theta}$ on the unit and counit. We start with the counit to prove that :

$$\epsilon_{H^*}\widetilde{\theta}(m\otimes\delta_g)=\epsilon_H(m\otimes\delta_g).$$

So

$$\begin{aligned}\epsilon_{H^*}\widetilde{\theta}(m\otimes\delta_g) &= \epsilon_{H^*}(\delta_{\theta(m\rhd g)}\otimes\theta(m\lhd g))\\ &= \delta_{\theta(m\rhd g),e}.\end{aligned}$$

To have a non-zero answer we should have $\theta(m \triangleright g) = e$, or $\theta(m \triangleright g) = \theta(e)$ which implies that $m \triangleright g = e$ since θ is invertible. Applying $m^L \triangleright$ to both sides gives

$$\begin{aligned} m^L \triangleright m \triangleright g &= m^L \triangleright e \\ \Rightarrow m^L m \triangleright g &= e \\ \Rightarrow e \triangleright g &= e \\ \Rightarrow g &= e, \end{aligned}$$

hence

$$\begin{aligned} \epsilon_{H^*}\widetilde{\theta}(m \otimes \delta_g) &= \epsilon_{H^*}(\delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g)) \\ &= \delta_{\theta(m \triangleright g),e} \\ &= \delta_{g,e} = \epsilon_H(m \otimes \delta_g). \end{aligned}$$

For the unit, we need to prove that $\widetilde{\theta}(1_H) = 1_{H^*}$, which we do as follows:

$$\begin{aligned} \widetilde{\theta}(1_H) &= \widetilde{\theta}(\sum_g e \otimes \delta_g) \\ &= \sum_{\theta(e \triangleright g)} \delta_{\theta(e \triangleright g)} \otimes \theta(e \triangleleft g) \\ &= \sum_{\theta(g)} \delta_{\theta(g)} \otimes \theta(e) \\ &= \sum_{\theta(g)} \delta_{\theta(g)} \otimes e = 1_{H^*}. \end{aligned}$$

To check that the antipode is preserved, we need the following calculations:

$$\begin{aligned} (mg)^L &= g^L m^L = g^{-1} m^L = ((m \triangleright g)(m \triangleleft g))^L \\ &= (m \triangleleft g)^L (m \triangleright g)^L = (m \triangleleft g)^L (m \triangleright g)^{-1} \\ &= ((m \triangleleft g)^L \triangleright (m \triangleright g)^{-1})((m \triangleleft g)^L \triangleleft (m \triangleright g)^{-1}). \end{aligned}$$

By the uniqueness of factorization, we should have

$$g^L = g^{-1} = ((m \triangleleft g)^L \triangleright (m \triangleright g)^{-1}), \tag{20}$$

and

$$m^L = ((m \triangleleft g)^L \triangleleft (m \triangleright g)^{-1}). \tag{21}$$

Due to the fact that θ is a semigroup isomorphism, we have

$$\theta(g^L) = \theta(g^{-1}) = (\theta(g))^L = \theta((m \triangleleft g)^L \triangleright (m \triangleright g)^{-1}), \tag{22}$$

$$\theta(m^L) = (\theta(m))^L = (\theta(m))^{-1} = \theta((m \lhd g)^L \lhd (m \rhd g)^{-1}). \tag{23}$$

Moreover, it is needed to prove that the antipode S satisfying

$$\widetilde{\theta}S(m \otimes \delta_g) = S\widetilde{\theta}(m \otimes \delta_g), \tag{24}$$

which we do as follows:

$$\begin{aligned}
\widetilde{\theta}S(m \otimes \delta_g) &= \widetilde{\theta}(S(m \otimes \delta_g)) \\
&= \widetilde{\theta}((m \lhd g)^L \otimes \delta_{(m \rhd g)^{-1}}) \\
&= \delta_{\theta((m \lhd g)^L \rhd (m \rhd g)^{-1})} \otimes \theta((m \lhd g)^L \lhd (m \rhd g)^{-1}) \\
&= \delta_{(\theta(g))^L} \otimes (\theta(m))^{-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
S\widetilde{\theta}(m \otimes \delta_g) &= S(\widetilde{\theta}(m \otimes \delta_g)) \\
&= S(\delta_{\theta(m \rhd g)} \otimes \theta(m \lhd g)) \\
&= \delta_{(\theta(m \rhd g) \lhd \theta(m \lhd g))^L} \otimes (\theta(m \rhd g) \rhd \theta(m \lhd g))^{-1} \\
&= \delta_{(\theta(g))^L} \otimes (\theta(m))^{-1},
\end{aligned}$$

as required.

Finally, to see that $\widetilde{\theta} : H^* \to H$ is invertible, we define $\widetilde{\theta}^{-1} : H^* \to H$ by

$$\widetilde{\theta}^{-1}(\delta_m \otimes g) = \theta^{-1}(m \rhd g) \otimes \delta_{\theta^{-1}(m \lhd g)}, \tag{25}$$

and we want to prove that :

$$\widetilde{\theta}\widetilde{\theta}^{-1}(\delta_m \otimes g) = id(\delta_m \otimes g),$$

and

$$\widetilde{\theta}^{-1}\widetilde{\theta}(m \otimes \delta_g) = id(m \otimes \delta_g),$$

where id is the identity map.

$$\begin{aligned}
\widetilde{\theta}\widetilde{\theta}^{-1}(\delta_m \otimes g) &= \widetilde{\theta}\big(\widetilde{\theta}^{-1}(\delta_m \otimes g)\big) \\
&= \widetilde{\theta}\big(\theta^{-1}(m \rhd g) \otimes \delta_{\theta^{-1}(m \lhd g)}\big) \\
&= \delta_{\theta(\theta^{-1}(m \rhd g) \rhd \theta^{-1}(m \lhd g))} \otimes \theta(\theta^{-1}(m \rhd g) \lhd \theta^{-1}(m \lhd g)) \\
&= \delta_{\theta(\theta^{-1}(m))} \otimes \theta\theta^{-1}(g) \\
&= \delta_m \otimes g.
\end{aligned}$$

Also,

$$\begin{aligned}
\widetilde{\theta}^{-1}\widetilde{\theta}(m \otimes \delta_g) &= \widetilde{\theta}^{-1}(\widetilde{\theta}(m \otimes \delta_g)) \\
&= \widetilde{\theta}^{-1}(\delta_{\theta(m \rhd g)} \otimes \theta(m \lhd g)) \\
&= \theta^{-1}(\theta(m \rhd g) \rhd \theta(m \lhd g)) \otimes \delta_{\theta^{-1}(\theta(m \rhd g) \lhd \theta(m \lhd g))} \\
&= \theta^{-1}(\theta(m)) \otimes \delta_{\theta^{-1}(\theta(g))} \\
&= m \otimes \delta_g,
\end{aligned}$$

as required. Therefore, $\widetilde{\theta}$ is a left Hopf algebra isomorphism. ■

Proposition 2.4 *Let $H = kM \bowtie k(G)$ be a left Hopf algebra associated to a factorization of a group $X = GM$ into a subgroup G and a subsemigroup M whit identity and a left inverse property where $k(G)$ is the Hopf algebra of function on the subgroup G and kM is the semigroup left Hopf algebra of the semigroup M . Then the we can induce the factor-reversing semigroup isomorphism of $X = GM$ from a left Hopf algebra self-duality pairings $<,>$: $H \otimes H \to k$ on the left Hopf algebra H. The formula for the corresponding pairing is*

$$< m \otimes \delta_g, m_1 \otimes \delta_{g_1} >= \delta_{m,\theta(m_1 \triangleright g)} \delta_{g,\theta(m_1 \triangleleft g_1)}. \tag{26}$$

Proof. Assume that $\widetilde{\theta} : H \to H^*$ which is defined by

$$\widetilde{\theta}(m \otimes \delta_g) = \delta_{\theta(m \triangleright g)} \otimes \theta(m \triangleleft g),$$

is a left Hopf algebra isomorphism which sends basis elements of H to basis elements of H^*. We want to prove that we can induce a semigroup isomorphism θ from $\widetilde{\theta}$. We start with functions $\mathfrak{m} : M \times G \to M$ and $\mathfrak{g} : M \times G \to G$ such that

$$\widetilde{\theta}^{-1}(\delta_m \otimes g) = \mathfrak{m}(m, g) \otimes \delta_{\mathfrak{g}(m,g)}. \tag{27}$$

The condition that $\widetilde{\theta}^{-1}$ preserves the unit gives

$$\widetilde{\theta}^{-1}(1_{H^*}) = \widetilde{\theta}^{-1}(\sum_m \delta_m \otimes e) = \sum_{\mathfrak{g}(m,e)} \mathfrak{m}(m, e) \otimes \delta_{\mathfrak{g}(m,e)}.$$

But

$$\widetilde{\theta}^{-1}(1_{H^*}) = \widetilde{\theta}^{-1}(\sum_m \delta_m \otimes e) = \sum_g e \otimes \delta_g = 1_H,$$

since $\widetilde{\theta}^{-1}$ is an algebra isomorphism. From these we see that

$$\mathfrak{m}(m, e) = e. \tag{28}$$

Also, the preservation of the counit gives

$$\epsilon_H \widetilde{\theta}^{-1}(\delta_s \otimes u) = \epsilon_H(\mathfrak{m}(m, g) \otimes \delta_{\mathfrak{g}(m,g)}) = \delta_{\mathfrak{g}(m,g),e}.$$

But

$$\epsilon_H \widetilde{\theta}^{-1}(\delta_m \otimes g) = \epsilon_{H^*}(\delta_m \otimes g) = \delta_{m,e},$$

since $\widetilde{\theta}^{-1}$ is a coalgebra isomorphism. Putting $m = e$, we find

$$\mathfrak{g}(e, g) = e. \tag{29}$$

Next, we use the fact that $\widetilde{\theta}^{-1}$ is an algebra homomorphism in the equation

$$\widetilde{\theta}^{-1}((\delta_m \otimes g)(\delta_{m_1} \otimes g_1)) = \widetilde{\theta}^{-1}(\delta_m \otimes g)\widetilde{\theta}^{-1}(\delta_{m_1} \otimes g_1), \tag{30}$$

to obtain the following equivalent equations:

$$\widetilde{\theta}^{-1}(\delta_{m\triangleleft g,m_1}(\delta_m \otimes gg_1)) = (\mathfrak{m}(m,g) \otimes \delta_{\mathfrak{g}(m,g)})(\mathfrak{m}(m_1,g_1) \otimes \delta_{\mathfrak{g}(m_1,g_1)}),$$

$$\delta_{m\triangleleft g,m_1}\widetilde{\theta}^{-1}(\delta_m \otimes gg_1) = \delta_{\mathfrak{g}(m,g),\mathfrak{m}(m_1,g_1)\triangleright\mathfrak{g}(m_1,g_1)}(\mathfrak{m}(m,g)\mathfrak{m}(m_1,g_1) \otimes \delta_{\mathfrak{g}(m_1,g_1)}),$$

or

$$\delta_{m\triangleleft g,m_1}(\mathfrak{m}(m,gg_1) \otimes \delta_{\mathfrak{g}(m,gg_1)}) = \delta_{\mathfrak{g}(m,g),\mathfrak{m}(m_1,g_1)\triangleright\mathfrak{g}(m_1,g_1)}(\mathfrak{m}(m,g)\mathfrak{m}(m_1,g_1) \otimes \delta_{\mathfrak{g}(m_1,g_1)}).$$

To have a non-zero answer we should have :

$$m_1 = m \triangleleft g, \tag{31}$$

and

$$\mathfrak{g}(m,g) = \mathfrak{m}(m_1,g_1) \triangleright \mathfrak{g}(m_1,g_1). \tag{32}$$

Thus, for all $m, m_1 \in M$ and $g, g_1 \in G$, we deduce

$$\mathfrak{m}(m,gg_1) = \mathfrak{m}(m,g)\mathfrak{m}(m_1,g_1), \tag{33}$$

$$\mathfrak{g}(m,gg_1) = \mathfrak{g}(m_1,g_1). \tag{34}$$

Note that if we put $m = e$ in (33) and substitute $m_1 = m \triangleleft g$, we get

$$\mathfrak{m}(e,gg_1) = \mathfrak{m}(e,g)\mathfrak{m}(e,g_1). \tag{35}$$

Now, the equations for preservation of the coproduct yield

$$\begin{aligned}\Delta\widetilde{\theta}^{-1}(\delta_m \otimes g) &= \Delta(\mathfrak{m}(m,g) \otimes \delta_{\mathfrak{g}(m,g)})\\ &= \sum_{x,y\in G, xy=\mathfrak{g}(m,g)} \mathfrak{m}(m,g) \otimes \delta_x \otimes (\mathfrak{m}(m,g) \triangleleft x) \otimes \delta_y,\end{aligned}$$

and

$$\begin{aligned}\Delta\widetilde{\theta}^{-1}(\delta_m \otimes g) &= (\widetilde{\theta}^{-1} \otimes \widetilde{\theta}^{-1})\Delta(\delta_m \otimes g) \quad (\text{since } \widetilde{\theta}^{-1} \text{ is a colagebra isomorphism})\\ &= (\widetilde{\theta}^{-1} \otimes \widetilde{\theta}^{-1})\Big(\sum_{a,b\in M: ab=m} \big(\delta_a \otimes (b \triangleright g)\big) \otimes \big(\delta_b \otimes g\big)\Big)\\ &= \sum_{a,b\in M: ab=m} \widetilde{\theta}^{-1}\big(\delta_a \otimes (b \triangleright g)\big) \otimes \widetilde{\theta}^{-1}\big(\delta_b \otimes g\big)\\ &= \sum_{a,b\in M: ab=m} \mathfrak{m}(a, b \triangleright g) \otimes \delta_{\mathfrak{g}(a,b\triangleright g)} \otimes \mathfrak{m}(b,g) \otimes \delta_{\mathfrak{g}(b,g)}.\end{aligned}$$

Thus

$$\mathfrak{g}(m,g)=\mathfrak{g}(a,b\triangleright g)\mathfrak{g}(b,g)=\mathfrak{g}(ab,g).$$

Putting $g=e$ gives

$$\mathfrak{g}(m,g)=\mathfrak{g}(a,b\triangleright e)\mathfrak{g}(b,e)=\mathfrak{g}(a,e)\mathfrak{g}(b,e)=\mathfrak{g}(ab,e). \tag{36}$$

From the coproduct formula we also see that $\mathfrak{m}(m,g)\triangleleft x=\mathfrak{m}(b,g)$ where $x=\mathfrak{g}(a,b\triangleright g)$ and $ab=m$. Putting $b=e$ here gives

$$\mathfrak{m}(m,g)\triangleleft\mathfrak{g}(a,g)=\mathfrak{m}(e,g).$$

From (36), we have $\mathfrak{g}(m,g)=\mathfrak{g}(ab,g)$. Putting $b=e$ gives $\mathfrak{g}(m,g)=\mathfrak{g}(a,g)$. Hence

$$\mathfrak{m}(m,g)\triangleleft\mathfrak{g}(m,g)=\mathfrak{m}(e,g). \tag{37}$$

From (32), with $m=m_1\triangleleft g^{-1}$ and $g=e$, we get:

$$\begin{aligned}\mathfrak{m}(m_1,g_1)\triangleright\mathfrak{g}(m_1,g_1)&=\mathfrak{g}(m,g)\\&=\mathfrak{g}(m_1\triangleleft g^{-1},g)\\&=\mathfrak{g}(m_1\triangleleft g^{-1},e)\\&=\mathfrak{g}(m_1\triangleleft e,e)\quad(\text{as } g=e,\text{ then } g^{-1}=e,\ g\in G)\\&=\mathfrak{g}(m_1,e).\end{aligned}$$

Consequently,

$$\mathfrak{m}(m_1,g_1)\triangleright\mathfrak{g}(m_1,g_1)=\mathfrak{g}(m_1,e). \tag{38}$$

If we put (37) and (38) together, then

$$\mathfrak{m}(m,g)\mathfrak{g}(m,g)=(\mathfrak{m}(m,g)\triangleright\mathfrak{g}(m,g))(\mathfrak{m}(m,g)\triangleleft\mathfrak{g}(m,g))=\mathfrak{g}(m,e)\mathfrak{m}(e,g). \tag{39}$$

From (34) with $g_1=e$, we have

$$\mathfrak{g}(m,g)=\mathfrak{g}(m_1,e).$$

Also, from (31), we have $m_1=m\triangleleft g$ which implies that

$$\mathfrak{g}(m,g)=\mathfrak{g}(m\triangleleft g,e).$$

From the coproduct formula we get $\mathfrak{m}(a,b\triangleright g)=\mathfrak{m}(m,g)$ and $m=ab$ which implies that

$$\mathfrak{m}(a,b\triangleright g)=\mathfrak{m}(ab,g).$$

Putting $a = e$ gives

$$\mathfrak{m}(e, b \triangleright g) = \mathfrak{m}(b, g).$$

Substiuting these equations into (39) gives

$$\mathfrak{m}(e, m \triangleright g)\mathfrak{g}(m \triangleleft g, e) = \mathfrak{g}(m, e)\mathfrak{m}(e, g). \quad (40)$$

Therefore, equations (28), (29), (35), (36), and (40) are all the conditions needed to prove that the map $\psi : X \to X$ defined by

$$\begin{aligned}\psi(mg) &= \mathfrak{g}(m, e)\mathfrak{m}(e, g)\\ &= \psi(m)\psi(g).\end{aligned}$$

is a semigroup homomorphism. Since $G \bigcap M = e$, the map ψ is well defined. If we set $\theta = \psi^{-1}$ we see that our original left Hopf algebra map $\widetilde{\theta}$ is indeed that induced by θ. ■

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References

[1] M. M. Al-Shomrani and E. J. Beggs, *Making nontrivially associated modular categories from finite groups. Int. J. Math. and Math. Science, 2004 (42) (2004), 2231-2264.*

[2] M. M. Al-Shomrani, *Algebras and their dual in rigid tensor categories, Int. Math. Forum, 1 (9 -12)(2006) 525 - 550.*

[3] J. Beachy, *Introductory Lectures on Rings and Modules. Cambridge University Press,(1999), 26-27 .*

[4] E. J. Beggs, J. D. Gould and S. Majid, *Finite group factorizations and braiding. J. Algebra, 181 (1) (1996) 112 - 151.*

[5] E. J. Beggs and S. Majid, *Quasitriangular and differential structures on bicrossproduct Hopf algebras. J. Algebra, 219 (2) (1996) 682 - 727.*

[6] J. A. Green, W. D. Nichols and E. J. Taft, *Left Hopf algebras. J. Algebra, 65 (2) (1980) 399 - 411.*

[7] S. Majid, *Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. J. Algebra, 130 (1990) 17 - 64.*

[8] S. Majid, *The quantum double as quantum mechanics. J. Geom. Phys., 13 (1994) 169 - 202.*

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